PHYSICAL REVIEW E **77**, 047601 (2008)

Suppression of the generation of defect modes by a moving soliton in an inhomogeneous Toda lattice

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(Received 28 December 2007; published 29 April 2008)

The motion of solitons is studied in the Toda lattice with a local defect due to a change in coupling constants. We demonstrate that the generation of the trapped defect mode by the incident soliton is strongly suppressed under a certain condition. The effect is explained by the fact that, under this condition, the defect mode vanishes in the linear limit. In the same case, the soliton remains stable, traveling through a periodic array of defects; otherwise, it decays.

DOI: 10.1103/PhysRevE.77.047601 PACS number(s): 05.45.Yv, 42.81.Dp

I. INTRODUCTION AND THE MODEL

The dynamics of nonlinear lattices is a subject of great interest in its own right, and serves as a basis for modeling a large number of physical objects, such as crystals, polymer molecules, arrays of optical waveguides, Bose-Einstein condensates (BECs) trapped in deep optical lattices, arrays of coupled Josephson junctions, etc. (see, e.g., recent review [1] and topical collection of articles [2]). Many lattice systems support collective excitations in the form of solitons and cnoidal waves (as widely adopted in physics literature, using the word "soliton," we do not refer to integrable systems).

The Toda lattice (TL) is a well-known integrable model, which provides exact solutions for moving solitons in the lattice [3]. A generalized (inhomogeneous, hence nonintegrable) TL model is based on the chain equations of motion, $m_n\ddot{u}_n = F_n - F_{n+1}$, where the overdot stands for the time derivative, m_n and u_n are the mass and displacement of the nth particle, and the interaction forces are

$$F_n(r_n) = a_n[\exp(-b_n r_n) - 1], \tag{1}$$

with a_n and b_n constants of the nth nonlinear spring, and $r_n \equiv u_n - u_{n-1}$ relative displacements. In terms of r_n, the equations of motion take the form of

$$m_n\ddot{r}_n = 2F_n - F_{n+1} - F_{n-1}. \tag{2}$$

In the uniform (integrable) TL, with $m_n \equiv m$, $a_n \equiv a$, and $b_n \equiv b$, rescaling makes it possible to set $a = b = m \equiv 1$ in Eq. (2), which we assume below, as concern the uniform part of the TL in the present model.

Realistic lattice models may also include local imperfections, viz., impurities, i.e., particles with a different mass, and spring defects, featuring local variations of constants a and b. Interaction of solitons with impurities in the TL was studied by Nakamura and Takeno [4], and later considered in other works [5]. Other types of imperfections in the TL were studied too, including surface defects [6] and interfaces between two lattices with different parameters [7]. Various effects generated by the impurities and defects have been reported, such as bounce and fission of incident solitons, excitation of localized defect modes, etc. The interaction of lattice solitons with imperfections may play an important role in specific physical models. In particular, these studies were recently extended to various defects in photonic/optical lattices, in the context of BEC and nonlinear optics [8]. Of special interest is the interaction of gap solitons in fiber Bragg gratings with attractive defects in the form of a short fiber segment with suppressed Bragg reflectivity [9], and cavities formed by a pair of repulsive defects, created as short segments with enhanced reflectivity [10].

In this Brief Report, we consider the scattering of the lattice soliton on a local defect in the TL described by Eq. (2) with equal masses, which is accounted for by a variation of the elastic constants for nonlinear springs linking the pairs of particles $(-1,0)$ and $(0,1)$, as shown in Fig. 1,

$$(a,b)_n = 1 + [(a^*,b^*) - 1](\delta_{n,0} + \delta_{n,1}). \tag{3}$$

A similar model (with unequal masses) was introduced by Nakamura [4], in the cases of strong and weak defects, which corresponds to $a^*, b^* > 1$ and $a^*, b^* < 1$, respectively.

Results are reported in the next section. They demonstrate the existence of a curve in the plane of parameters a^* and b^* along which the local defect is "transparent," i.e., the collision of the soliton with the defect does not generate a bound mode. The existence of this curve is explained in terms of the linearized model. We also consider circulation of the soliton in a ring-shaped TL with periodic boundary conditions and an embedded defect, and demonstrate that, under the same "transparency" condition, the circulating soliton maintains its shape, while in other cases it decays due to collision-induced losses.

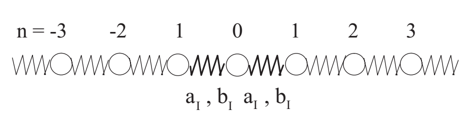

FIG. 1. The Toda lattice with the spring defect.

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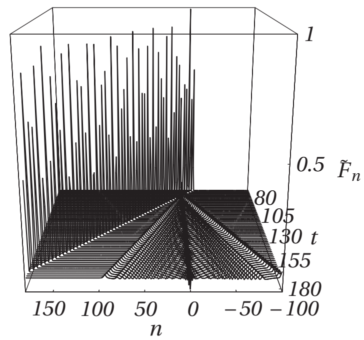


FIG. 2. The scattering of the soliton on the lattice defect with $a^*=2.0$ and $b^*=1.4$.

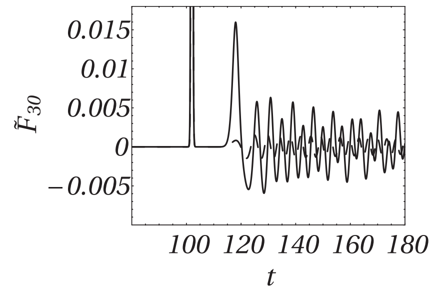


FIG. 4. The lattice perturbation versus time at $n=30$, for $a^*=2.0$, and $b^*=1.4$ or $b^*=0.797$ (the full and dashed lines, respectively).

II. NUMERICAL AND ANALYTICAL RESULTS

To simulate Eq. (2), we used an explicit Runge-Kutta method of the ninth order, with an embedded error estimator. The TL was composed of 400 particles, with periodic boundary conditions. The initial condition was taken as per the exact soliton in the uniform TL (with $a=b=m=1$, as said above),

$$r_n^{(0)} = -\ln\left(1 + \frac{\sinh^2(k_0)}{\cosh^2[k_0(n-n_0)]}\right), \tag{4}$$

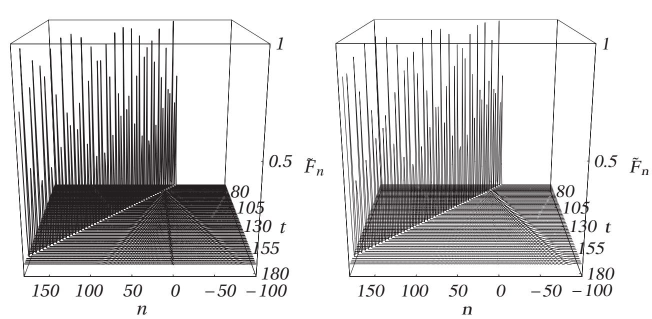
with the center placed at $n_0=-189$. The amplitude of the unperturbed soliton and its velocity, $v=k_0^{-1}\sinh(k_0)$ (we chose $v>0$), are determined by parameter k_0. First, we produce results for a characteristic value of this parameter, $k_0=2.3$, and then present results for a range of other values of k_0.

The simulations demonstrate that, after the passage of the soliton through the defect, a small-amplitude trapped mode emerges, along with transmitted and reflected "radiation" waves. A typical example is displayed in Fig. 2, by means of a set of stacked plots for $\tilde{F}_n \equiv F_n/\sinh^2(k_0)$ versus n, each one pertaining to a given moment of time [recall F_n is the local force defined in Eq. (1)]. The fluctuations observed in the soliton amplitude in Fig. 2 are due to the fact that the soliton is very narrow, hence its maximum does not necessarily coincide with a lattice site.

The generation of the trapped mode and scattered radiation, as a result of the collision of moving TL solitons with coupling and mass impurities (both weak and strong ones), were already demonstrated in Refs. [4,5]. A feature that our simulations reveal is that one can find "antiresonant" values of the defect parameters, at which the generation of the trapped defect mode, as well as scattered radiation, are strongly suppressed. For instance, fixing $a^*=2.0$, we find that the trapped defect mode is generated with a very small amplitude at $b^*=0.797$, as shown in Fig. 3(a). Additionally, Fig. 3(b) shows that parameter values can be found at which the forward-scattered radiation as well as the trapped defect mode are strongly reduced (although they do not vanish), while backscattering is slightly enhanced. The suppression effect can also be observed by plotting $\tilde{F}_n$ versus time at a fixed lattice site, as shown in Fig. 4.

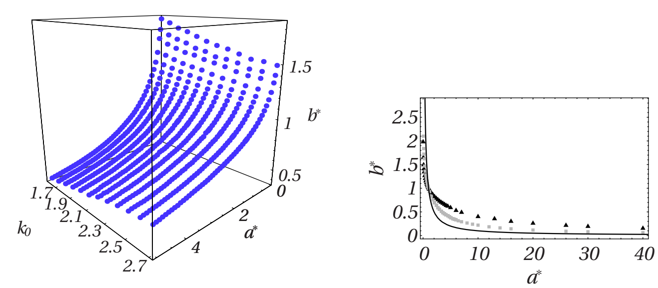
Systematic analysis demonstrates that, for a given amplitude k_0 of incident soliton (4) (i.e., for a fixed velocity of the soliton), there is a specific relation between coefficients a^* and b^* providing for the strongest suppression of the excitation of the defect mode. In Fig. 5(a), stacked plots show this relation, for different values of k_0. The numerical data make it possible to identify the defect-mode-suppression condition in a sufficiently sharp form, therefore uncertainty in the relation are included in the finite size of the data points.

Point $a^*=b^*=1$ corresponds to the strongest suppression, as in this case there is no defect, therefore all curves in Fig.



(a) (b)

FIG. 3. The scattering of the soliton with $k_0=2.3$ on the local defect with (a) $a^*=2.0$, $b^*=0.797$, and (b) $a^*=2.0$, $b^*=0.85$. In the former case, the defect mode is generated with a very small amplitude, while in the latter case it is practically absent, while a conspicuous backscattered wave is observed.



(a) (b)

FIG. 5. (Color online) (a) Values of parameters a^* and b^* at which the generation of the defect mode is suppressed, for fixed values of k_0, as indicated in the figure. (b) Curves from (a) for $k_0=1.6$ (squares) and $k_0=2.7$ (triangles). Hyperbola $a^*b^*=1$ is predicted by the analytical consideration (see text).

5 pass through this point. To understand the effect in the presence of the defect, we use the expansion of Eqs. (2) and (1), up to the quadratic terms (taking into regard that $m_n \equiv 1$),

$$\ddot{r}_n = a_{n+1}b_{n+1}r_{n+1} + a_{n-1}b_{n-1}r_{n-1} - 2a_n b_n r_n - (a_{n+1}b_{n+1}^2 r_{n+1}^2/2 + a_{n-1}b_{n-1}^2 r_{n-1}^2/2 - a_n b_n^2 r_n^2). \quad (5)$$

The substitution of expressions (3) in Eq. (5) demonstrates that, in the linear approximation, which corresponds to the first line in Eq. (5), parameters a^* and b^* appear (in equations for $n=-1,0,1$, and 2) only in the form of combination a^*b^*. Therefore, the defect mode, that should be found from the linearized version of Eqs. (5), depends solely on this combined parameter. Because the defect mode does not exist for $a^*=b^*=1$, it is obvious that the "transparency" line in parameter plane (a^*,b^*), at which the defect mode does not exist, amounts to hyperbola $a^*b^*=1$. To test this prediction, in Fig. 5(b) we compare it with actual curves from Fig. 5(a), taken for cases of weak and strong nonlinearity (small and large k_0, respectively). Naturally, the numerical data for the weaker nonlinearity fall closer to the hyperbola predicted by the linearization.

The deviation from the hyperbola can be estimated from the quadratic terms in Eq. (5). Indeed, the deviation is tangible for $a^*>1$, when the defect's nonlinearity may be appreciable. In this case, in agreement with Fig. 5(b), Eq. (5) predicts that the nonlinearity should shift the "transparency" curve to larger values of a^*b^*. The size of the shift is expected to be proportional to the soliton's amplitude. Indeed, for $k_0^{(1)}=2.7$ and $k_0^{(2)}=1.6$, the TL-soliton solution (4) yields the amplitude ratio $2\ln[\cosh(k_0^{(1)}/k_0^{(2)})]\approx 2.13$, which is consistent with Fig. 5(b).

A straightforward extension of the soliton-defect collision is passage of a soliton through a periodic array of defects, which was simulated with the periodic boundary conditions and a single defect placed at $n=0$. The plots displayed in Fig. 6 demonstrate that the soliton's amplitude is reduced after each collision, unless the defect's parameters belong to "transparency" set (a^*,b^*), as defined above. If the parameters belong to this set, the soliton travels across the array of defects indefinitely long without any conspicuous loss.

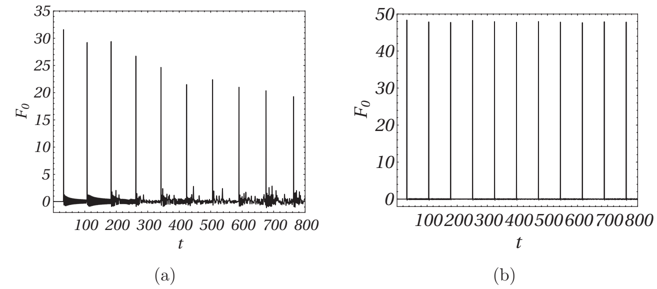


FIG. 6. The motion of a soliton in the ring lattice with 200 sites and the defect set at $n=0$, for $a^*=2.0$, $k_0=2.7$, and $b^*=0.5$ (a) or $b^*=0.833$ (b). In the latter case, the parameters belong to the set which provides for the suppression of inelastic effects in the collision of the soliton with the individual defect, as per Fig. 5.

III. CONCLUSIONS

In the Toda lattice with a local defect represented by modified values of the nonlinear-spring constants, we have identified a parameter subspace in which the generation of the defect mode due to the collision of the incident soliton is strongly suppressed. A simple explanation to this fact was given, as a relation between parameters at which the defect mode vanishes in the linear limit. In this "transparency" subspace, the amplitude of the soliton traveling through an array of defects remains virtually constant; otherwise, it decays due to collision-induced losses. We expect that a similar effect may be observed in nonintegrable lattice models where, strictly speaking, traveling solitons do not exist, but may persist on a long time scale, which makes the consideration of soliton-defect collisions a relevant issue.

ACKNOWLEDGMENTS

L.V. appreciates discussions with Y. Kubota and T. Odagaki and comments from R. Labbé. This work was partially supported by DICYT-USACH under Grant No. 2070700.

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